Other tests (optional)

Convergence Tests

1) <u>Comparison Test</u>

If $\boxed{a_n \geq b_n \geq 0}$, then

a) If $\sum_{n=1}^{\infty} a_n$ converges,

then $\sum_{n=1}^{\infty} b_n$ converges.

b) If $\sum_{n=1}^{\infty} b_n$ diverges, then

$\sum_{n=1}^{\infty} a_n$ diverges

2) <u>Limit comparison Test</u>

If $a_n \geq 0$ and $b_n > 0$,

then if

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \underline{L > 0},$$

(L is a number)

then either both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge or both diverge.

The interval of convergence of a power series is just all real numbers for which the series converges.

General Power Series

Of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

c = the center of the series

= the number that makes every term zero

The radius of convergence is always $\frac{1}{2}$ the length of the interval of convergence.

If R = radius of convergence

c = center, the interval

of convergence is always

$$(c-R,\ c+R)$$

and maybe the endpoints

$x = c-R,\ x = c+R$

Example 1: (center, radius, interval)

$$\sum_{n=0}^{\infty} \frac{(x-4)^n}{(3n^3+4)5^n}$$

Find the center, radius, & interval of convergence.

$C = 4$

Ratio test for radius first

$$a_n = \frac{(x-4)^n}{(3n^3+4)5^n}, \quad a_{n+1} = \frac{(x-4)^{n+1}}{(3(n+1)^3+4)5^{n+1}}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x-4}{5}\cdot\frac{3n^3+4}{3(n+1)^3+4}\right|$$

$$= \frac{|x-4|}{5}\,\frac{3n^3+4}{3(n+1)^3+4}$$

$$\lim_{n\to\infty}\frac{|x-4|}{5}\cdot\frac{3n^3+4}{3(n+1)^3+4}$$

$$= \frac{|x-4|}{5}\lim_{n\to\infty}\frac{3n^3+4}{3(n+1)^3+4}$$

$$\overset{\text{L'H}}{=} \frac{|x-4|}{5}\lim_{n\to\infty}\frac{\cancel{9}n^2}{\cancel{9}(n+1)^2} = 1$$

$$= \frac{|x-4|}{5}\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^2 = \frac{|x-4|}{5}$$

$$\left| \frac{x-4}{5} \right| < 1$$

is where ratio test tells us the series converges.

This is

$$-1 < \frac{x-4}{5} < 1$$

multiply by 5

$$-5 < x-4 < 5$$

add 4

$$-1 < x < 9 \quad (R = 5)$$

½ length of interval

Plug in $x=-1$ and $x=9$

into the original series

Series is $$\sum_{n=0}^{\infty} \frac{(x-4)^n}{(3n^3+4)5^n}$$

$x=9$ $$\sum_{n=0}^{\infty} \frac{(9-4)^n}{(3n^3+4)5^n}$$

$$= \sum_{n=0}^{\infty} \frac{\cancel{5^n}}{(3n^3+4)\cancel{5^n}}$$

$$= \sum_{n=0}^{\infty} \frac{1}{3n^3+4}$$

Use limit comparison
with $a_n = \frac{1}{n^3}$, $b_n = \frac{1}{3n^3+4}$

We know $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges, (p-rule)

and $\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{3n^3+4}{n^3}$

$= 3 > 0.$

This shows $\sum_{n=0}^{\infty} \frac{1}{3n^3+4}$ converges

so $x=a$ is in the interval,

$x = -1$

$$\sum_{n=0}^{\infty} \frac{(-1-4)^n}{(3n^3+4)\,5^n}$$

$$= \sum_{n=0}^{\infty} \frac{(-5)^n}{(3n^3+4)5^n}$$ $= (-1)^n \cdot 5^n$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \cancel{5^n}}{(3n^3+4)\cancel{5^n}}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{3n^3+4}$$

$$\lim_{n\to\infty} \frac{1}{3n^3+4} = 0$$

and the sequence $\left(\frac{1}{3n^3+4}\right)_{n=1}^{\infty}$

decreases, so by the

Alternating Series Test

the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{3n^3+4}$

converges. So $x=-1$

is in the interval.

So: The interval of convergence is $[-1, 9]$, radius is 5, center is 4 (midpoint of the interval).

Power Series and Functions

$$\sum_{n=1}^{\infty} x^n = \frac{x}{1-x} \quad \text{if } |x|<1.$$

let's use this.

Example 2: Find a power series representation for $\frac{1}{1+x^2}$ on a nonzero radius of convergence.

$$\sum_{n=1}^{\infty} x^n = \frac{x}{1-x},$$ so dividing by x $(x \neq 0)$

$$\sum_{n=1}^{\infty} x^{n-1} = \frac{1}{1-x}$$ ($m = n-1$ and reindex to 0)

$$\sum_{m=0}^{\infty} x^m = \frac{1}{1-x}$$

Plug in $-x^2$ for x

on both sides

$$\sum_{m=0}^{\infty} (-x)^{2m} = \frac{1}{1+x^2}$$

Here's a power series,

what is the radius of

convergence?

ratio test

$$a_{m+1} = (-x)^{2(m+1)} = (-x)^{2m+2}$$

$$a_m = (-x)^{2m}$$

$$\left|\frac{a_{m+1}}{a_m}\right| = \left|\frac{(-x)^{2m+2}}{(-x)^{2m}}\right|$$

$$= |(-x)^2| = x^2$$

need $x^2 < 1$ $\leftrightarrow$ same as $|x| < 1$ to converge,

so $-1 < x < 1$ and $R = 1$.

Plug $x = \pm 1$ back in

$x = 1$

$$\sum_{m=0}^{\infty} (-1)^{2m}$$

$$= \sum_{m=0}^{\infty} 1^{2m} = \sum_{m=0}^{\infty} 1$$

Diverges by test for divergence.

So $x = 1$ is not in the interval.

$\underline{x = -1}$

$$\sum_{n=0}^{\infty} 1^{2n} = \sum_{n=0}^{\infty} 1$$

again Diverges, so $x = -1$ is not in the interval

So: Interval of convergence

$(-1, 1)$

Radius $= 1$

Center $= 0$

$$\frac{1}{1+x^2} = \sum_{m=0}^{\infty} (-x)^{2m}$$

This is $\displaystyle\sum_{m=0}^{\infty} a_m x^m$

for some sequence $(a_m)_{m=1}^{\infty}$

Find a_0, a_1, a_2, a_3.

Write out both series!

$$\sum_{m=0}^{\infty}(-x)^{2m} = 1 + 0 \cdot x + 1 \cdot x^2 + 0 \cdot x^3 + \cdots$$

$$\sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$a_0 = 1$

$a_1 = 0$ (no x^1 power in 1st series)

$a_2 = 1$

$a_3 = 0$ (no x^3 power in 1st series)

Differentiation and Integration

If $\sum_{n=0}^{\infty} a_n (x-c)^n$ converges

on some nonzero radius of convergence about $x=c$, then

if $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$, then

1) $f'(x) = \sum_{n=0}^{\infty} n \cdot a_n \cdot (x-c)^{n-1}$

2) $\int f(x)dx = \sum_{n=0}^{\infty} \frac{a_n (x-c)^{n+1}}{n+1} + C$

on the open interval of convergence

Example 3: $(\arctan(x))$

We know $\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x)^{2n}$

from previous example when $|x| < 1$. Integrate!

$$\arctan(x) = \int \frac{1}{1+x^2}\,dx = \int \sum_{n=0}^{\infty} (-x)^{2n}\,dx$$

$$= \int \sum_{n=0}^{\infty} x^{2n}\,dx$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} + C$$

$\arctan(0) = 0, \quad \text{so} \quad C = 0.$

$$\arctan(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}$$

$$= x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots$$

$$= 0 + x + 0 \cdot x^2 + \frac{x^3}{3} + 0 \cdot x^4 + \cdots$$

$$\arctan(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$a_0 = 0, \; a_1 = 1, \; a_2 = 0, \; a_3 = \frac{1}{3}, \; a_4 = 0$

$a_5 = 1/5, \; \cdots$

Example 4. $(\ln(x))$

We need a power series centered at some nonzero number since $\ln(0)$ does not exist!

Let's try $c = 1$ for our center.

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x} = \frac{1}{1-1+x}$$

$\neq 0$

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

$r = 1 - x$

$$= \frac{1}{1-(1-x)}$$

$$= \sum_{n=0}^{\infty} (1-x)^n$$

$$\ln(x) = \int \frac{d}{dx}(\ln(x))\,dx$$

$$= \int \sum_{n=0}^{\infty} (1-x)^n \,dx$$

$$= \int \sum_{n=0}^{\infty} (-1)^n (x-1)^n \,dx$$

$$= \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{n+1}}{n+1}}$$